MMP Learning Seminar.

Week 75.

Introduction to the theory of complements.

Theory of complements:

$\overline{\mathbb{K}} = \mathbb{K}$, char $\mathbb{K} = 0$, $\mathbb{Q}$-divisors

Definition: (X, Δ) log pair $X \xrightarrow{\varphi} Z$ proj contraction. Let $z \in Z$ a closed point.

A $\mathbb{Q}$-complement around z. is an effective $B \geq \Delta$ for which:

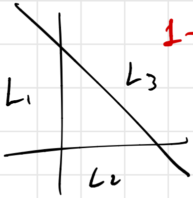
1) (X, B) has log canonical sing over a neighborhood of $z \in Z$.
2) $K_X + B \equiv_Z 0$ on a neighborhood of $z \in Z$.

In this case we say that (X, Δ) is $\mathbb{Q}$-complemented over the preimage of a neighborhood of $z \in Z$.

Rmk: $K_X + B \equiv_Z 0 \implies K_X + B \sim_{\mathbb{Q}, Z} 0$ (by work of Fujino and Gongyo).

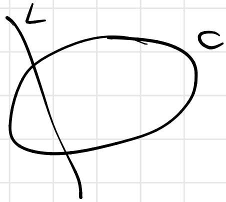
Example: $Z = pt$, $\Delta = 0$. A $\mathbb{Q}$-complement is nothing else than an effective divisor $0 \leq B \sim_{\mathbb{Q}} -K_X$ for which (X, B) has log canonical sing, i.e., log CY structure on X.

Example: $\mathbb{P}^2 \longrightarrow pt.$ $\mathbb{Q}$- complement.



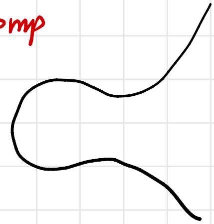
1-comp.

$(\mathbb{P}^2, L_1 + L_2 + L_3)$



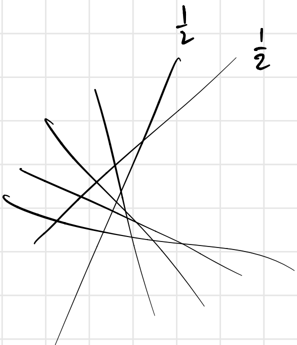
1-Comp

$(\mathbb{P}^2, L + C)$



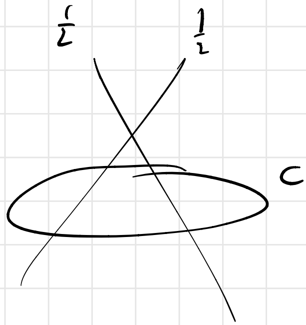
1-comp

$(\mathbb{P}^2, E)$



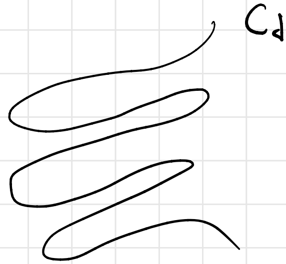
$(\mathbb{P}^2, \frac{1}{2}L_1 + \cdots + \frac{1}{2}L_6)$

2- comp.



$(\mathbb{P}^2, \frac{1}{2}L_1 + \frac{1}{2}L_2 + C)$

2- Comp.



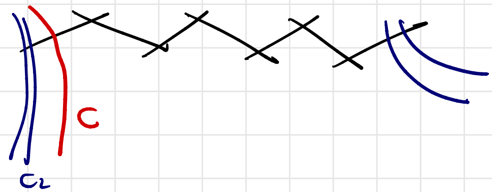
$(\mathbb{P}^2, \frac{3}{d} C_d)$

d- comp.

Example: $X \longrightarrow Z$, $x \in X$ a closed.

Then a $\mathbb{Q}$-complement is the structure of a lc sing around x

1-comp.

Example: $(D_n; 0) \xleftarrow{\ \pi\ }$ 

2-comp

$(D_n, C; 0)$ strictly lc, $\mathbb{Q}$-complement.

$(D_n, \frac{1}{2}C_1 + \frac{1}{2}C_2; 0)$ strictly lc, $\mathbb{Q}$-complement.

2-comp.

Definition: Let (X,Δ) be a log pair & $X \longrightarrow Z$ be a projective contraction.
Let N be a positive integer & $z \in Z$ a closed point.
We say that $B \geq 0$ on X is a N-complement over $z \in Z$.
if the following conditions are satisfied:

i) (X,B) is log canonical over a neighborhood of $z \in Z$.
ii) $N(K_X+B) \sim_Z 0$ after possibly shrinking around $z \in Z$.
iii) $NB \geq N\lfloor \Delta \rfloor + \lfloor (N+1)\Delta \rfloor$ ← Diophantine approx.

If $NB \geq N\Delta$, then we say it is a monotone N-comp.

$$\mathcal{S} = \left\{ 1 - \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$$

Rmk 1: $Z = pt$, $\Delta = 0$, N-complement, is an element of $|-NK_X|$ with nice sing.

Rmk 2: $X \longrightarrow Z$ identity and $x \in X$.
A N-complement is the structure of a lc sing with prescribed index

Conj: Let $\Lambda \subseteq \mathbb{Q}$, $\bar{\Lambda} \subseteq \mathbb{Q}$ and satisfies the DCC.
Let n be a positive integer. There exists $N := N(n, \Lambda)$
that satisfies the following.
Let (X, Δ) be a log pair of dimension n & $X \longrightarrow Z$ proj contr
$\mathrm{coeff}(\Delta) \subseteq \Lambda$. & $z \in Z$ a closed point
If (X, Δ) is $\mathbb{Q}$-complemented over $z \in Z$, then
(X, Δ) is N-complemented over $z \in Z$.

Phrasing: If a variety admits a lCY structures
then we can find it in a effective way
(in terms of $\dim X$)

$|-K_X|$, $|-2K_X|$, $|-3K_X|$, ...

Remark: $X \to Z$ Fano type, i.e.,

There exists $\Delta \geq 0$ for which $-(K_X+\Delta)$

nef & big over Z & (X, Δ) is klt.

$$|-m(K_X+\Delta)/Z| \ni \Gamma$$

$(X, \Delta + \Gamma/m)$ is a $\mathbb{Q}$-complement over Z.

Example: $\boxed{n=1, \quad \Lambda = \mathcal{S}}$, $Z = pt.$ $\longrightarrow$ $N=6.$

E , $K_E \sim 0.$

1-comp

$(\mathbb{P}^1, 1-\frac{1}{n}\{0\} + 1-\frac{1}{m}\{0\}) \longrightarrow (\mathbb{P}^1, \{0\} + \{\infty\})$

$(\mathbb{P}^1, \{0\} + \frac{1}{2}\{1\} + \frac{1}{2}\{\infty\}) \longrightarrow$ 2-comp

$(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{1}{2}\{1\} + \frac{1}{2}\{2\} + \frac{1}{2}\{\infty\}) \longrightarrow$ 2-comp

$(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{3}{4}\{1\} + \frac{4}{5}\{\infty\}) \longrightarrow (\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{3}{4}\{1\} + \frac{5}{6}\{\infty\})$

6-comp.

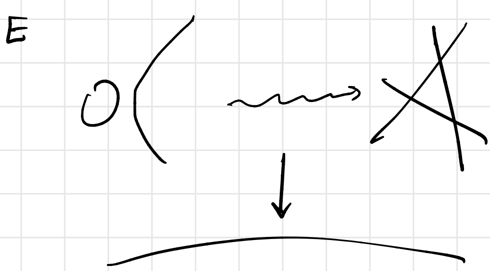
- n-dimensional Fano varieties can be "effectively" turned into CY pairs.

- X smooth CY of dim n, then we can find a flat degeneration $\mathcal{X} \longrightarrow \mathbb{A}^1$ for which $\mathcal{X}_t \simeq X$, $\mathcal{X}_0$ is snc with n comp intersecting non-trivially.
  ($\mathcal{D}(\mathcal{X}_0)$ is $(n-1)$-dimensional).

$\longrightarrow$ $\mathcal{X}$ is a maximal unipotent degeneration

$\longrightarrow$ $\mathcal{X}$ is a coregularity zero degeneration

Any component of $\mathcal{X}_0$ is birational to a Fano type var, in particular, any comp is rationally connected



$$\text{Fano type} \underset{\text{degenerations.}}{\overset{\text{complements}}{\rightleftarrows}} \text{Calabi-Yau}$$

Proposition 1: (Y,Δ) a log pair.

$Y \xrightarrow{\varphi} X$ proj bir morphism.

If $B_Y \geq \Delta$ is a $\mathbb{Q}$-comp (resp. monotone $\mathbb{N}$-comp), then $B = \varphi_* B_Y$ is a $\mathbb{Q}$-comp (resp. monotone $\mathbb{N}$-comp) of X.

Proof: B_Y is a monotone $\mathbb{N}$-comp of (Y,Δ):

i) (Y,Δ_Y) is log canonical, &

ii) $N(K_Y+\Delta_Y) \sim 0$.

$$\varphi^*(K_X+B) = K_Y + B_Y + \overbrace{E - F}$$

effective divisors, both φ-exceptional, and they have no common comp.

$$E - F \sim_{\mathbb{Q}} \varphi^*(K_X+B)$$

$$E - F \sim_{\mathbb{Q},X} 0.$$

$\varphi_*(E-F) = 0$, neg Lemma $\Longrightarrow$ $E \geq F$

neg Lemma $\Longrightarrow$ $F \geq E$

$\Big\}\; E = F = 0.$

$$\varphi^*(K_X+B) = K_Y + B_Y$$

is log canonical.

$N(K_Y+B_Y) \sim 0$, $\quad \varphi_* N(K_Y+B_Y) \sim 0 \quad N(K_X+B) \sim 0 \quad \square$

Proposition 2: (X,Δ) log canonical pair.

Let $X \overset{\varphi}{\dashrightarrow} X'$ be a sequence of steps of the $(-(K_X+\Delta))$-MMP. Assume (X',Δ') is lc, where $\Delta' =: \varphi_* \Delta$. Let (X',B') be a N-comp of (X',Δ'). Then (X,Δ) admits a N-comp.

Proof:

$$\begin{tikzcd} Y \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed] & X' \end{tikzcd}$$

$$p^*(K_X+\Delta) = q^*(K_{X'}+\Delta') - F$$

In particular,

$$a_E(X',\Delta') \leq a_E(X,\Delta).$$

$$q^*(K_{X'}+B') = K_Y + B_Y.$$

(Y, B_Y) sub-lc & $N(K_Y+B_Y) \sim 0$

Set $B = p_* B_Y$.

(X,B) is sub-lc & $N(K_X+B) \sim 0$

B is a boundary $\iff$ $a_E(X,B) \in [0,1]$ for all $E \subseteq X$.

$$1 \geq a_E(X,\Delta) \geq a_E(X',\Delta') \geq a_E(X',B') = a_E(X,B) \geq 0$$

$\square$

Strategy for the conj:

X Fano variety of dim n.

this is implied by BAB conj.

1) Look at all $\mathbb{Q}$-complements of X.

all exc Fanos of dim n

If all of them are klt, then is called an exceptional Fano. In this case, we expect that belong to a bounded family.

$(X,B) \rightsquigarrow (X,(1+\varepsilon)B)$ $\quad K_X+(1+\varepsilon)B \sim_{\mathbb{Q}} \varepsilon B$

$\sim_{\mathbb{Q}} -\varepsilon K_X$

big

nice coeff

remain klt.

2) (X,B) $\mathbb{Q}$-comp which is strictly lc.

$(Y, B_Y + \underbrace{E_1+\dots+E_k}_{S})$ dlt modification.

It suffices to produce a N-comp of

$(Y, \underbrace{E_1+\dots+E_k}_{S})$ by Prop 1.

(Y,S)

$(Y, B_Y + S)$ dlt & $K_Y + B_Y + S \equiv 0$

$(Y, (1+\varepsilon)B_Y + S)$ dlt.

log pair.

$$K_Y + (1+\varepsilon)B_Y + S \sim_{\mathbb{Q}} \varepsilon B_Y \sim_{\mathbb{Q}} -\varepsilon(K_Y + S)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

MMP $\qquad\qquad\qquad\qquad$ MMP

$$\begin{tikzcd} Y \arrow[r, dashed] & Y' \arrow[d] \\ & Z \end{tikzcd}$$

(Y', S')

By Prop 2 is enough to complement this.

$-(K_{Y'} + S')$ is semiample

Replace $X \rightsquigarrow (Y', S')$

Assume (X,S) is lc, $-(K_X+S)$ semiample.

$X \longrightarrow Z$ ample model.

2.1) $\dim X = \dim Z$, $\quad -(K_X+S)$ semiample $+$ big

2.2) $\dim Z = 0$, $\quad K_X + S \equiv 0$.

2.3) $\dim Z \in \{1, \dots, \dim X - 1\}$.

how to control the coeff?

$$\begin{array}{c} X \\ \varphi \downarrow \\ Z \end{array} \qquad K_X + S \sim_{\mathbb{Q}} \varphi^*(K_Z + \underbrace{B_Z + M_Z}_{\text{coefficients are controlled by coeff } S \text{ \& } \dim X})$$

$$N(K_X+S) \sim 0$$

$$\Uparrow$$

$$H^0(N(K_X+S)) \neq 0$$

$$\underset{\substack{\neq \\ 0}}{H^0(N(K_X+S))} \twoheadrightarrow \overset{\substack{0 \\ \neq}}{H^0(N(K_S+\Delta_S))}$$

2.1) $\dim X = \dim Z$, $\quad -(K_X+S)$ semiample + big

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may contract all div with coeff 1 (!)

$-(K_X+S)$ is ample (lose $\mathbb{Q}$-factorial).

$H^1(-K_X-2S)=0 \implies$

$$H^0(-(K_X+S)) \twoheadrightarrow H^0(-(K_S+\Delta_S)).$$

log Fano

By induction on the dim this has a N^1-comp.

$$H^0(-N(K_X+S)) \twoheadrightarrow H^0(-N(K_S+\Delta_S)).$$

$N^1(\dim X-1,\ \operatorname{coeff}\Delta_S)$.

only depends on $\dim X$

only depends on Δ $\quad \mathcal{D}(\Lambda)$

$$\mathcal{D}(\Lambda) = \{1-\lambda/m \mid \lambda\in\Lambda,\ m\in\mathbb{N}\}$$

Main Thm of comp: From global-to-local

$(Y, \Delta_Y + S)$ plt

$\downarrow$

$(X, \Delta; x)$

plt blow-ups

techniques from MMP

klt sing of dim n

$\mathrm{coeff}(\Delta) \subseteq \mathcal{S} = \{1 - \frac{1}{m} \mid m \in \mathbb{N}\}$.

$(K_Y + \Delta_Y + S)$ anti-ample over X

Δ_Y strict transform of Δ.

$$K_Y + \Delta_Y + S|_S = K_S + \Delta_S$$

log Fano of dim $n-1$

+

some MMP results

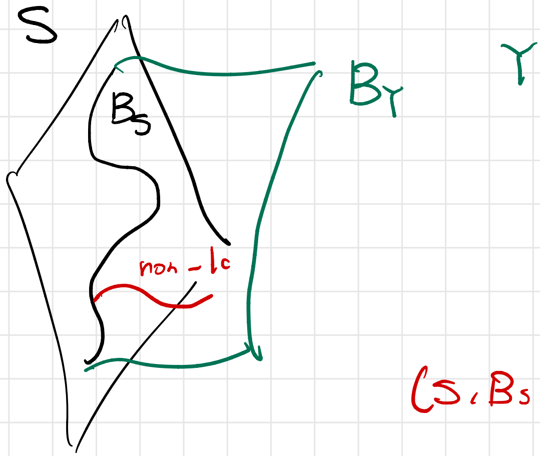
$N(K_S + B_S) \sim 0$

$$H^0(-(K_Y + \Delta_Y + S)) \longrightarrow H^0(-N(K_S))$$

$B_Y \subset$ (in $H^0(-(K_Y+\Delta_Y+S))$), $0 \leq B_S \subset$ (in $H^0(-N(K_S))$)

$N(K_Y + B_Y + \Delta_Y + S) \sim 0$

Q: Is it log canonical?



(S, B_S) not lc

$$Y \big\downarrow X$$

$\left\{ \begin{array}{l} (Y, B_Y + \Delta_Y + S) \text{ is lc} \\ \text{around } S \end{array} \right\}$

$\Big\{$ **Thm:** (X, B) pair & $-(K_X + B)$ ample
The non-klt locus is connected $\Big\}$

$\square$

Second main theorem on complements.

Theorem 1.4: Assume BAB conjecture in dim n.

(boundeds of Fano varieties with mild singularities)

$+$

Effective canonical bundle formula conj. in dim n

$\Downarrow$

Existence of bounded complements in dim n.

Redo 2.1 + 2.2, they explain how to use the cbf to solve 2.3.

2.3) $\dim Z \in \{1, \dots, \dim X - 1\}$.

$$\begin{array}{c} X \\ \varphi \downarrow \\ Z \end{array} \qquad K_X + S \sim_{\mathbb{Q}} \varphi^*(K_Z + B_Z + M_Z)$$

X Fano type $\Longrightarrow$ Z Fano type

$M_Z \geq 0$

Find a N-complement $\Gamma_Z \geq 0$ of $(Z, B_Z + M_Z)$

$N(K_Z + B_Z + M_Z + \Gamma_Z) \sim 0$ & $(Z, B_Z + M_Z + \Gamma_Z)$ is lc

$\boxed{\Gamma_X := \varphi^*(\Gamma_Z)}$ → coefficients of Γ_X are controlled by coeff of Γ_Z

$K_X + S + \Gamma_X \equiv 0$.

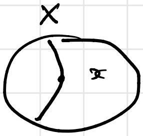
$N(K_X + S + \Gamma_X) \sim 0$. $\Longleftarrow$ Step 2.2

$(X, S + \Gamma_X)$ is lc. $\Longleftarrow$ $K_X + S + \Gamma_X \sim_{\mathbb{Q}} \varphi^*(K_Z + B_Z + M_Z + \Gamma_Z)$.

(preserves sing.)

$\square$

$(X; x)$ sing.



$(X, B; x)$ $\quad N(K_X+B) \sim 0.$

$\Updownarrow$ index one cover of K_X+B has degree N.

K_X+B is Cartier on X^{sm}

$$\pi_1(X^{sm}; x) \twoheadrightarrow \mathbb{Z}_N.$$